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Journal of Geometry and Physics 50 (2004) 162–187

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Noncommutative two-tori with real multiplication as noncommutative projective varieties

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Received 12 May 2003; accepted 18 November 2003

Abstract

We define analogues of homogeneous coordinate algebras for noncommutative two-tori with real multiplication. We prove that the categories of standard holomorphic vector bundles on such noncommutative tori can be described in terms of graded modules over appropriate homogeneous coordinate algebras. We give a criterion for such an algebra to be Koszul and prove that the Koszul dual algebra also comes from some noncommutative two-torus with real multiplication. These results are based on the techniques of [Categories of holomorphic bundles on noncommutative two-tori. math.AG/0211262] allowing to interpret all the data in terms of autoequivalences of the derived categories of coherent sheaves on elliptic curves.

Keywords: Noncommutative two-tori; Koszul dual algebra; Morita autoequivalence

1. Introduction

Noncommutative algebraic geometry is usually understood as the study of certain Abelian categories replacing the usual category of (quasi-)coherent sheaves (see [2,10,16]). For example, noncommutative projective schemes correspond to certain categories defined in terms of modules over graded algebras in the way analogous to Serre's theorem (see [2]). However, it is rather disappointing that at present there is almost no connection between noncommutative algebraic varieties over $\mathbb{C}$ and noncommutative topological spaces, which according to Connes [6] are described by C^*-algebras. One of the indications that such a connection exists is provided by the work [7], where Sklyanin algebras are related to some noncommutative manifolds. In the present paper we give another example of a relation of this kind. Namely, we show that noncommutative two-tori admitting "real multiplication"

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0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2003.11.007

(i.e., nontrivial Morita autoequivalences) can be considered as underlying noncommutative topological spaces for certain noncommutative algebraic varieties.

This relation is not so surprising given the recent studies of complex geometry on noncommutative two-tori (see [8,14,17]). It may only seem a little odd that real multiplication is relevant for our picture. Let us briefly explain this. Recall that the homogeneous coordinate algebra of a projective scheme is defined using tensor powers of an ample line bundle. In noncommutative world one can only take tensor powers of a bimodule. Therefore, in order to construct an analogue of such algebra for a noncommutative two-torus T_θ, where $\theta \in \mathbb{R}$, one has to find a bimodule over the ring of functions on T_θ which would be ample in appropriate sense. The natural choice would be one of the so-called basic modules. Now the Morita equivalence theory for noncommutative two-tori implies that an interesting bimodule can be found among basic modules only when the parameter θ is stabilized by a nontrivial element of $\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ under the fractional-linear action of this group on $\mathbb{R} \cup \{\infty\}$. In other words, the category of vector bundles on T_θ should have a nontrivial Morita autoequivalence. Note however, that there exists a generalization of the standard approach to noncommutative projective schemes in which graded algebras are replaced by more general objects called $\mathbb{Z}$-algebras (see [4,19]). If one allows these more general noncommutative "$\mathbb{Z}$-projective schemes" then the condition that T_θ has real multiplication becomes unnecessary.

The results of this paper depend heavily on the study of categories of holomorphic vector bundles on T_θ in [14]. Recall that in *loc. cit.* we considered only certain class of holomorphic bundles on T_θ that we called *standard* and we constructed a fully faithful functor from the category of such bundles to the derived category $D^b(X)$ of coherent sheaves on some elliptic curve X. Moreover, we proved that the image of this functor consists of stable objects in the heart $\mathcal{C}^\theta$ of certain nonstandard t-structure on $D^b(X)$ associated with θ (see 1.2; these t-structures were defined in [5]). We conjecture that every holomorphic bundle on T_θ is a successive extension of standard holomorphic bundles. If true, this would imply an equivalence of $\mathcal{C}^\theta$ with the category of all holomorphic bundles on T_θ (for irrational θ). Since we do not know the validity of this conjecture, we simply replace the category of all holomorphic bundles on T_θ by $\mathcal{C}^\theta$. This allows us to switch from the context of noncommutative complex geometry on T_θ to the study of the t-structure on $D^b(X)$ associated with θ. Nontrivial Morita autoequivalences of T_θ appearing when θ is a quadratic irrationality correspond to nontrivial autoequivalences $F : D^b(X) \to D^b(X)$ preserving the corresponding t-structure.

The graded algebras associated with T_θ can now be viewed as examples of the following general construction. Given an additive category $\mathcal{C}$, an additive functor $F : \mathcal{C} \to \mathcal{C}$ and an object O of $\mathcal{C}$, we define an associative graded ring

$$A_{F,O} = \bigoplus_{n\geq 0} \mathrm{Hom}_{\mathcal{C}}(O, F^n(O)),$$

where $(F^n : \mathcal{C} \to \mathcal{C}, n \geq 0)$ are the functors obtained by iterating F (so $F^0 = \mathrm{Id}_{\mathcal{C}}$). The multiplication is defined as the composition of the natural maps

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}}(O, F^m(O)) \otimes \mathrm{Hom}_{\mathcal{C}}(O, F^n(O))\\ &\to \mathrm{Hom}_{\mathcal{C}}(F^n(O), F^{m+n}(O)) \otimes \mathrm{Hom}_{\mathcal{C}}(O, F^n(O)) \to \mathrm{Hom}_{\mathcal{C}}(O, F^{m+n}(O)).\end{aligned}$$

The homogeneous coordinate ring of a projective scheme X appears as a particular case of this construction when $\mathcal{C}$ is the category of coherent sheaves on X, F is the functor of tensoring with an ample line bundle L on X, $O = \mathcal{O}_X$ is the structure sheaf. Slightly more general rings are obtained when taking F to be of the form $F(A) = L \otimes \sigma^* A$, where σ is an automorphism of X. The corresponding rings are twisted homogeneous coordinate rings considered in [1].

The example relevant for noncommutative tori with real multiplication is when $\mathcal{C} = \mathcal{C}^\theta \subset D^b(X)$, where X is an elliptic curve, F is the autoequivalence of $D^b(X)$ preserving $\mathcal{C}^\theta$. In Section 3 we study corresponding graded algebras $A_{F,\mathcal{F}}$, where $\mathcal{F}$ is a stable object of $\mathcal{C}^\theta$. Namely, we compute the Hilbert series of $A_{F,\mathcal{F}}$ and formulate simple criterions in terms of discrete invariants of $(F, \mathcal{F})$ for the algebra $A_{F,\mathcal{F}}$ to be generated in degree 1, to be quadratic, and to be Koszul. We also observe that if $A_{F,\mathcal{F}}$ is Koszul then the Koszul dual algebra is again of the same form: it is isomorphic to $A_{R_{\mathcal{F}}\circ F^{-1},\mathcal{F}}$, where $R_{\mathcal{F}}$ is certain twist functor associated with $\mathcal{F}$ (see Section 3.3).

In Section 4 we prove that every category $\mathcal{C}^\theta$, where θ is a quadratic irrationality, contains an *ample* sequence of objects of the form $(F^n\mathcal{F})$, where $F : \mathcal{C}^\theta \to \mathcal{C}^\theta$ is an autoequivalence. This means that $\mathcal{C}^\theta$ can be recovered from the corresponding graded algebra $A_{F,\mathcal{F}}$ by the noncommutative analogue of Proj-construction considered in [2]. One technical point is that the categories $\mathcal{C}^\theta$ are non-Noetherian, so we have to apply the main result of [13] that generalizes (a part of) the main theorem of [2] to non-Noetherian case.

It would be interesting to try to extend some of our results to more general algebras of the form $A_{F,\mathcal{F}}$, where F is an autoequivalence of the derived category $D^b(X)$ of coherent sheaves on a smooth projective variety X, $\mathcal{F}$ is an object of $D^b(X)$. The first natural question is whether there are interesting examples when F preserves some t-structure on $D^b(X)$. In the case when X is an Abelian variety one source of such examples should be given by noncommutative tori generalizing the picture described in [14].

Another perspective for the future work is to try to connect our results with Manin's program in [11] to use noncommutative two-tori with real multiplication for the explicit construction of the maximal Abelian extensions of real quadratic fields.

Convention. With the exception of Section 3.1 all the objects (varieties, categories) are defined over an arbitrary field k.

2. Preliminaries on derived categories of elliptic curves

2.1. Structure of the group of autoequivalences

Let X be an elliptic curve, $\mathrm{Aut}(D^b(X))$ be the group of (isomorphism classes of) exact autoequivalences of $D^b(X)$. There is a natural surjective homomorphism

$$\pi : \mathrm{Aut}(D^b(X)) \to \mathrm{SL}_2(\mathbb{Z}),$$

defined by the rule $\pi(F) = g \in \mathrm{SL}_2(\mathbb{Z})$, such that for every object $\mathcal{F} \in D^b(X)$ one has

$$\begin{pmatrix} \deg F(\mathcal{F}) \\ \mathrm{rk}\, F(\mathcal{F}) \end{pmatrix} = g \begin{pmatrix} \deg \mathcal{F} \\ \mathrm{rk}\, \mathcal{F} \end{pmatrix}.$$

For example, if F is the functor of tensoring with a line bundle L then F projects to the matrix

$$\begin{pmatrix} 1 & \deg(L) \\ 0 & 1 \end{pmatrix}.$$

Let $\mathcal{S} : D^b(X) \to D^b(X)$ be the Fourier–Mukai transform considered as an autoequivalence of $D^b(X)$ via the isomorphism $\hat{X} \simeq X$. Then

$$\pi(\mathcal{S}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The shift functor $\mathcal{F} \mapsto \mathcal{F}[1]$ maps under π to the matrix $-\mathrm{id} \in \mathrm{SL}_2(\mathbb{Z})$. We denote by $\mathrm{Aut}(X)$ the group of automorphisms of X preserving the neutral element. It can be identified with a subgroup of $\mathrm{Aut}(D^b(X))$: to every automorphism $\sigma : X \to X$ there corresponds an autoequivalence $\sigma_* = (\sigma^{-1})^* : D^b(X) \to D^b(X)$. Clearly, the homomorphism π is trivial on this subgroup.

On the other hand, for every Abelian variety X there is a homomorphism

$$\gamma_X : \mathrm{Aut}(D^b(X)) \to \mathrm{Sp}(X \times \hat{X}),$$

where $\mathrm{Sp}(X \times \hat{X})$ is the group of *symplectic* automorphisms of $X \times \hat{X}$, i.e. automorphisms preserving the line bundle $p_{14}^*\mathcal{P} \otimes p_{23}^*\mathcal{P}^{-1}$ on $(X \times \hat{X})^2$, where $\mathcal{P}$ is the Poincaré line bundle on $X \times \hat{X}$. The homomorphism γ_X was defined by Orlov [12, Corollary 2.16]. He also proved that it fits into an exact sequence

$$1 \to (X \times \hat{X})(k) \times \mathbb{Z} \to \mathrm{Aut}(D^b(X)) \xrightarrow{\gamma_X} \mathrm{Sp}(X \times \hat{X}) \to 1,$$

where the subgroup $(X \times \hat{X})(k)$ corresponds to functors of translation by points of X and of tensor products with line bundles in $\mathrm{Pic}^0(X)$, while $\mathbb{Z} \subset \mathrm{Aut}(D^b(X))$ is the subgroup of shifts $A \mapsto A[n]$. More precisely, to a point $(x, \xi) \in (X \times \hat{X})(k)$ one associates the autoequivalence

$$\Phi_{(x,\xi)} : D^b(X) \to D^b(X) : \mathcal{F} \mapsto t_{-x}^*(\mathcal{F}) \otimes \mathcal{P}|_{X \times \xi},$$

where $t_{x'} : X \to X$ denotes the translation by $x' \in X(k)$. The subgroup $(X \times \hat{X})(k) \subset \mathrm{Aut}(D^b(X))$ is normal and the adjoint action of $F \in \mathrm{Aut}(D^b(X))$ is given precisely by $\gamma_X(F)$ (see [12, Corollary 2.13]).

In the case of an elliptic curve we can identify X with $\hat{X}$, so the group $\mathrm{Sp}(X \times \hat{X})$ can be identified with the group $\mathrm{Sp}(X \times X)$ of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with entries in the ring $\mathrm{End}(X)$ satisfying the equations

$$\bar{a}d - \bar{c}b = 1, \qquad a\bar{d} - b\bar{c} = 1, \qquad \bar{a}c = \bar{c}a,$$
$$\bar{b}d = \bar{d}b, \qquad a\bar{b} = b\bar{a}, \qquad c\bar{d} = d\bar{c},$$

where $f \mapsto \bar{f}$ is the Rosati involution on $\mathrm{End}(X)$.

Lemma 2.1. *The group* $\mathrm{Sp}(X \times X)$ *is isomorphic to* $(\mathrm{Aut}(X) \times \mathrm{SL}_2(\mathbb{Z}))/\{\pm 1\}$, *where the subgroup* $\{\pm 1\}$ *is embedded diagonally.*

Proof. Note that for every $a \in \mathrm{Aut}(X)$ one has $\bar{a}a = 1$. Hence, $\mathrm{Aut}(X)$ embeds into $\mathrm{Sp}(X \times X)$ as the central subgroup of diagonal matrices

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

We have to prove that every element in $\mathrm{Sp}(X \times X)$ is a product of such a matrix with a matrix in $\mathrm{SL}_2(\mathbb{Z})$. If one of the entries of a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(X \times X)$$

is zero then this is easy. Assuming that all the entries are nonzero we note that the condition $a\bar{b} \in \mathbb{Z}$ implies that $a \in \mathbb{Q}b$. Therefore, we can write $a = a'r$, $b = b'r$ for some $r \in \mathrm{End}(X)_{\mathbb{Q}}$ and a pair of relatively prime integers (a', b'). From the condition $a, b \in \mathrm{End}(X)$ we immediately derive that $r \in \mathrm{End}(X)$. Using the conditions $\bar{a}c \in \mathbb{Z}$, $\bar{b}d \in \mathbb{Z}$ we can also write $c = c'\bar{r}^{-1}$, $d = d'\bar{r}^{-1}$ for some rational numbers (c', d'). Since c, d, and $\bar{r}$ are elements of $\mathrm{End}(X)$ we obtain that c' and d' are integers. The equation $\bar{a}d - \bar{c}b = 1$ implies that the matrix

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

belongs to $\mathrm{SL}_2(\mathbb{Z})$. Finally, since $c'\bar{r}^{-1} \in \mathrm{End}(X)$ and $d'\bar{r}^{-1} \in \mathrm{End}(X)$ it follows that $\bar{r}^{-1} \in \mathrm{End}(X)$, so r is a unit. □

Let us set $\overline{\mathrm{Aut}}(D^b(X)) = \mathrm{Aut}(D^b(X))/(X \times \hat{X})(k)$. According to the above lemma the homomorphism γ_X induces a surjective homomorphism

$$\gamma_X : \overline{\mathrm{Aut}}(D^b(X)) \to (\mathrm{Aut}\, X \times \mathrm{SL}_2(\mathbb{Z}))/\{\pm 1\}$$

with the kernel $\mathbb{Z}$. The homomorphism π also factors through a homomorphism

$$\pi : \overline{\mathrm{Aut}}(D^b(X)) \to \mathrm{SL}_2(\mathbb{Z}).$$

It is easy to check that the homomorphisms $\overline{\mathrm{Aut}}(D^b(X)) \to \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ induced by γ_X and π differ by an automorphism of $\mathrm{SL}_2(\mathbb{Z})$. On the other hand, there is a natural action of $\mathrm{Aut}(D^b(X))$ on $K_0(X)$ that preserves the subgroup $K_0(X)_0 \subset K_0(X)$ consisting of classes of zero degree and zero rank. Note that the determinant gives an isomorphism $\det : K_0(X)_0 \to \mathrm{Pic}^0(X)$. From this one can see that the action of $\mathrm{Aut}(D^b(X))$ on $K_0(X)_0$ factors through $\overline{\mathrm{Aut}}(D^b(X))$.

Theorem 2.2. *There exists a homomorphism* $\overline{\mathrm{Aut}}(D^b(X)) \to \mathrm{Aut}(X) : F \mapsto \alpha_F$, *such that*

$$F(a) = \alpha_F(a)$$

for every $a \in K_0(X)_0$. The induced homomorphism

$$\overline{\mathrm{Aut}}(D^b(X)) \to \mathrm{Aut}(X) \times \mathrm{SL}_2(\mathbb{Z}) : F \mapsto (\alpha_F, \pi(F))$$

fits into the exact sequence

$$1 \to 2\mathbb{Z} \to \overline{\mathrm{Aut}}(D^b(X)) \to \mathrm{Aut}(X) \times \mathrm{SL}_2(\mathbb{Z}) \to 1,$$

where $2\mathbb{Z}$ is the subgroup of even shifts $A \mapsto A[2n]$.

Proof. We have an exact sequence

$$1 \to \mathbb{Z} \to \overline{\mathrm{Aut}}(D^b(X))/\mathrm{Aut}(X) \xrightarrow{\bar{\pi}} \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\} \to 1, \tag{2.1}$$

where the homomorphism $\bar{\pi}$ is induced by π, $\mathbb{Z}$ is the subgroup of shifts. Indeed, this follows from the fact that $\bar{\pi}$ differs from the homomorphism induced by γ_X by an automorphism of $\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ and from the fact that $\ker(\gamma_X) = \mathbb{Z}$. One consequence of this is that $\overline{\mathrm{Aut}}(D^b(X))$ is generated by the subgroup $\mathrm{Aut}(X)$ together with the shift $A \mapsto A[1]$, the tensoring functor $T(A) = A \otimes L$, where L is a line bundle of degree 1, and the Fourier–Mukai transform $\mathcal{S}$. Since the action of all of these autoequivalences on $K_0(X)$ is known, we derive that for every $F \in \overline{\mathrm{Aut}}(D^b(X))$ there exists an automorphism $\alpha_F \in \mathrm{Aut}(X)$ such that F acts on $K_0(X)_0$ in the same way as α_F. Similarly, we can consider the action of F on $K_0(X \otimes_k \bar{k})_0$, where $\bar{k}$ is an algebraic closure of k. The above argument shows that there exists unique α_F defined over k, such that F acts on $K_0(X \otimes_k \bar{k})_0$ as α_F. It is clear that the homomorphism $F \mapsto \alpha_F$ restricts to the identity map on $\mathrm{Aut}(X) \subset \overline{\mathrm{Aut}}(D^b(X))$. Together with surjectivity of π this immediately implies surjectivity of the map $F \mapsto (\alpha_F, \pi(F))$. The kernel of this map clearly contains the subgroup of even shifts $2\mathbb{Z} \subset \mathbb{Z} \subset \overline{\mathrm{Aut}}(D^b(X))$. Using the exact sequence (2.1) one can easily see that this kernel coincides with $2\mathbb{Z}$. □

Let $\overline{\mathrm{Aut}}(D^b(X))^0 \subset \overline{\mathrm{Aut}}(D^b(X))$ (resp., $\mathrm{Aut}(D^b(X))^0 \subset \mathrm{Aut}(D^b(X))$) be the subgroup consisting of F with $\alpha_F = 1$. From the above theorem we get an exact sequence

$$1 \to 2\mathbb{Z} \to \overline{\mathrm{Aut}}(D^b(X))^0 \to \mathrm{SL}_2(\mathbb{Z}) \to 1.$$

The following proposition can be viewed as an analogue of the theorem of the cube for autoequivalences of $D^b(X)$.

Proposition 2.3. *For every $F \in \mathrm{Aut}(D^b(X))$ one has*

$$[F^3(a)] - (N_F + \alpha_F)[F^2(a)] + (1 + N_F\alpha_F)[F(a)] - \alpha_F[a] = 0 \tag{2.2}$$

in $K_0(X)$ for every $a \in K_0(X)$, where $N_F = \mathrm{tr}(\pi(F))$. In particular, if $F \in \mathrm{Aut}(D^b(X))^0$ then

$$[F^3(a)] - (N_F + 1)[F^2(a)] + (N_F + 1)[F(a)] - [a] = 0.$$

Proof. Set $N = N_F$. We claim that $[F^2(a)] - N[F(a)] + [a] \in K_0(X)_0 \subset K_0(X)$ for every $a \in K_0(X)$. Indeed, this follows immediately from the fact that F acts as an element g on the pair (deg, rk), where $g^2 - Ng + 1 = 0$. It remains to apply F to this element in $K_0(X)_0$ and to use the definition of α_F. $\square$

Example 2.4. Let F be of the form $(\otimes L) \circ \sigma^*$, where L is a line bundle, σ is a translation by a point on X. Then $\alpha_F = 1$ and

$$F^2 \simeq (\otimes L \otimes \sigma^* L) \circ \sigma^*, \qquad F^3 \simeq (\otimes L \otimes \sigma^* L \otimes (\sigma^2)^* L) \circ \sigma^*,$$

so (2.2) amounts to the identity

$$[L \otimes \sigma^* L \otimes (\sigma^2)^* L] - 3[L \otimes \sigma^* L] + 3[L] - [\mathcal{O}_X] = 0,$$

or equivalently

$$(\sigma^2)^* L \simeq \sigma^* L^2 \otimes L^{-1}.$$

On the other hand, this isomorphism is a direct consequence of the theorem of the cube. This is why we can view Proposition 2.3 as an analogue of this theorem for more general autoequivalences.

2.2. *t-structures on* $D^b(X)$

Let us say that an object $\mathcal{F} \in D^b(X)$ is *stable* if $\mathcal{F} = V[n]$, where $n \in \mathbb{Z}$, V is either a stable vector bundle or the structure sheaf of a k-point. It is easy to see that an object $\mathcal{F} \in D^b(X)$ is stable if and only if $\mathrm{Hom}(\mathcal{F}, \mathcal{F}) = k$ (since every object in $D^b(X)$ is isomorphic to the direct sum of its cohomology sheaves).

Below we are going to use some basic notions and results of the torsion theory that can be found in [9]. For every real number θ we consider a t-structure $(D^{\theta,\le 0}, D^{\theta,\ge 0})$ on $D^b(X)$ defined as follows. First, let us define a torsion pair $(\mathrm{Coh}_{>\theta}, \mathrm{Coh}_{\le\theta})$ in the category $\mathrm{Coh}(X)$ of coherent sheaves on X. By the definition, $F \in \mathrm{Coh}_{>\theta}$ (resp., $\mathrm{Coh}_{\le\theta}$) if all semistable factors of F have slope $> \theta$ (resp., $\le \theta$). Note that objects of $\mathrm{Coh}_{>\theta}$ are allowed to have arbitrary torsion (we consider torsion sheaves as having the slope $+\infty$). Now the t-structure associated with θ is defined by the rule

$$D^{\theta,\le 0} := \{K \in D^b(X) : H^{>0}(K) = 0,\, H^0(K) \in \mathrm{Coh}_{>\theta}\},$$
$$D^{\theta,\ge 1} := \{K \in D^b(X) : H^{<0}(K) = 0,\, H^0(K) \in \mathrm{Coh}_{\le\theta}\}.$$

The fact that this is indeed a t-structure follows from the torsion theory (see [9]). The heart of this t-structure is $\mathcal{C}^\theta(X) := D^{\theta,\le 0} \cap D^{\theta,\ge 0}$. It is equipped with the torsion pair $(\mathrm{Coh}_{\le\theta}[1], \mathrm{Coh}_{>\theta})$.

It is convenient to extend the above definition to $\theta = \infty$ by letting $(D^{\infty,\le 0}, D^{\infty,\ge 0})$ to be the standard t-structure on $D^b(X)$. In the following proposition we list some properties of these t-structures. Let us denote $v_{\mathcal{F}} = (\deg(\mathcal{F}), \mathrm{rk}(\mathcal{F})) \in \mathbb{Z}^2$ for $\mathcal{F} \in D^b(X)$. Note that for $F \in \mathrm{Aut}(D^b(X))$ one has $v_{F(\mathcal{F})} = \pi(F)(v_{\mathcal{F}})$, where $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ as on column vectors.

Proposition 2.5.

(i) *The category* $\mathcal{C}^\theta(X)$ *has cohomological dimension* 1 *and there is an equivalence* $D^b(\mathcal{C}^\theta(X)) \simeq D^b(X)$.

(ii) *If* $\theta \in \mathbb{R} \setminus \mathbb{Q}$ *then one has*

$$\{v_{\mathcal{F}} | \mathcal{F} \in \mathcal{C}^{\theta}(X), \mathcal{F} \neq 0\} = H_{\theta} \cap \mathbb{Z}^2,$$

where $H_{\theta} = \{(x_1, x_2) | x_1 - \theta x_2 > 0\}$. *If* $\theta \in \mathbb{Q}$ *then the above statement is true with the half-plane* H_{θ} *replaced by its union with the ray* $\mathbb{R}_{\leq 0}(\theta, 1)$.

Proof.

(i) Let us first prove that $\mathrm{Hom}^{>1}_{D^b(X)}(A, B) = 0$ for every $A, B \in \mathcal{C}^{\theta}(X)$. If both objects A and B belong to one of the subcategories $\mathrm{Coh}_{>\theta}$ or $\mathrm{Coh}_{\leq\theta}[1]$, then the assertion is clear. If $A \in \mathrm{Coh}_{>\theta}$, $B \in \mathrm{Coh}_{\leq\theta}[1]$ then $\mathrm{Hom}^{i}_{D^b(X)}(A, B) = \mathrm{Hom}^{i+1}_{D^b(X)}(A, B[-1]) = 0$ for $i \geq 1$. On the other hand,

$$\mathrm{Hom}^{i}_{D^b(X)}(B, A) = \mathrm{Hom}^{i-1}_{D^b(X)}(B[-1], A) \simeq \mathrm{Hom}^{2-i}_{D^b(X)}(A, B[-1])^* = 0$$

for $i \geq 2$, since $\mathrm{Hom}_{\mathrm{Coh}(X)}(A, B[-1]) = 0$.

The second assertion follows from this by the standard argument (see, e.g. [3]).

(ii) This follows from the fact that primitive lattice vectors contained in $H \cup \mathbb{R}_{\leq 0}(\theta, 1)$ are exactly vectors $v_{\mathcal{F}}$, where $\mathcal{F}$ is a stable object belonging to $\mathrm{Coh}_{>\theta}$ or $\mathrm{Coh}_{\leq\theta}[1]$.

□

It is not difficult to calculate the action of autoequivalences of $D^b(X)$ on these t-structures. We denote by $\theta \mapsto g\theta = (a\theta + b)/(c\theta + d)$, where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

the standard fractional-linear action of g on $\mathbb{R} \cup \{\infty\}$.

Proposition 2.6. *For every* $F \in \mathrm{Aut}(D^b(X))$ *and every* $\theta \in \mathbb{R} \cup \{\infty\}$ *one has* $F(D^{\theta,\leq 0}) = D^{g\theta,\leq 0}[n]$ *(resp.,* $F(D^{\theta,\geq 0}) = D^{g\theta,\geq 0}[n]$*), where* $g = \pi(F) \in \mathrm{SL}_2(\mathbb{Z})$, *n is some integer*.

Proof. The assertion is clear when F is a translation or the tensor product with a line bundle, or the pull-back under an automorphism. Hence, it suffices to consider the case $F = \mathcal{S}$, where $\mathcal{S}$ is the Fourier–Mukai transform. For every segment $I \subset \mathbb{R} \cup \{+\infty\}$ let us denote by $\mathrm{Coh}_I(X) \subset \mathrm{Coh}(X)$ the full subcategory in $\mathrm{Coh}(X)$ consisting of sheaves F such that all semistable factors of F have slope in I (recall that torsion sheaves have slope $+\infty$). Assume first that $\theta \in \mathbb{R}$, $\theta > 0$. Since $\mathcal{S}$ transforms the slopes by the map $\mu \mapsto -\mu^{-1}$, we have

$$\mathcal{S}(\mathrm{Coh}_{(\theta,+\infty]}) = \mathrm{Coh}_{(-\theta^{-1},0]}, \qquad \mathcal{S}(\mathrm{Coh}_{(0,\theta]}[1]) = \mathrm{Coh}_{(-\infty,-\theta^{-1}]}[1],$$
$$\mathcal{S}(\mathrm{Coh}_{(-\infty,0]}[1]) = \mathrm{Coh}_{(0,+\infty]}.$$

This immediately implies that $\mathcal{S}$ sends the t-structure associated with θ to the t-structure associated with $-\theta^{-1}$. Since $\mathcal{S}^2 = [-\mathrm{id}]^*[-1]$, it follows that for $\theta \in \mathbb{R}$, $\theta < 0$ one has $\mathcal{S}(D^{\theta,\leq 0}, D^{\theta,\geq 0}) = (D^{-\theta^{-1},\leq 0}[-1], D^{-\theta^{-1},\geq 0}[-1])$. Similarly, it easy to see that $\mathcal{S}$ switches (up to a shift) the standard t-structure with the t-structure corresponding to $\theta = 0$. □

Let us consider the bilinear form $\chi(\mathcal{F}_1, \mathcal{F}_2) := \sum_i (-1)^i \dim \mathrm{Hom}^i(\mathcal{F}_1, \mathcal{F}_2)$ on $K_0(X)$, where $\mathrm{Hom}^i(\mathcal{F}_1, \mathcal{F}_2) := \mathrm{Hom}_{D^b(X)}(\mathcal{F}_1, \mathcal{F}_2[i])$. It is easy to see that

$$\chi(\mathcal{F}_1, \mathcal{F}_2) = -\det(v_{\mathcal{F}_1}, v_{\mathcal{F}_2}).$$

The kernel of χ is exactly the subgroup $K_0(X)_0 \subset K_0(X)$ consisting of elements of zero degree and zero rank. Abusing the notation we also set $\chi(v, v') := -\det(v, v')$ for $v, v' \in \mathbb{Z}^2$, so that $\chi(\mathcal{F}_1, \mathcal{F}_2) = \chi(v_{\mathcal{F}_1}, v_{\mathcal{F}_2})$.

The following lemma generalizes to categories $\mathcal{C}^\theta(X)$ the well-known fact about stable bundles on X.

Lemma 2.7. *Let $\mathcal{F}_1, \mathcal{F}_2$ be a pair of stable objects in $\mathcal{C}^\theta(X)$ such that $\chi(\mathcal{F}_1, \mathcal{F}_2) > 0$. Then* $\mathrm{Hom}^1(\mathcal{F}_1, \mathcal{F}_2) = 0$ *and* $\dim \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = \chi(\mathcal{F}_1, \mathcal{F}_2)$.

Proof. By Proposition 2.5(i) we have $\chi(\mathcal{F}_1, \mathcal{F}_2) = \dim \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) - \dim \mathrm{Hom}^1(\mathcal{F}_1, \mathcal{F}_2)$, so it is enough to check the vanishing of $\mathrm{Hom}^1(\mathcal{F}_1, \mathcal{F}_2)$. In the case when both $\mathcal{F}_1$ and $\mathcal{F}_2$ belong to $\mathrm{Coh}_{>\theta}$ (resp., $\mathrm{Coh}_{\leq\theta}[1]$) the assumption $\chi(v_{\mathcal{F}_1}, v_{\mathcal{F}_2}) > 0$ implies that $\mu(\mathcal{F}_1) < \mu(\mathcal{F}_2)$ (resp., $\mu(\mathcal{F}_1[-1]) < \mu(\mathcal{F}_2[-1])$). Hence, in this case the assertion is clear. It is easy to see that the case $\mathcal{F}_1 \in \mathrm{Coh}_{\leq\theta}[1]$, $\mathcal{F}_2 \in \mathrm{Coh}_{>\theta}$ cannot occur. Indeed, since the vectors $v_{\mathcal{F}_1}$ and $v_{\mathcal{F}_2}$ belong to H_θ and $\chi(v_{\mathcal{F}_1}, v_{\mathcal{F}_2}) > 0$, the condition $\mathrm{rk}\, \mathcal{F}_2 > 0$ implies that $\mathrm{rk}\, \mathcal{F}_1 > 0$. In the remaining case $\mathcal{F}_1 \in \mathrm{Coh}_{>\theta}$ and $\mathcal{F}_2 \in \mathrm{Coh}_{\leq\theta}[1]$, so the assertion follows from the vanishing of $\mathrm{Hom}^2(\mathcal{F}_1, \mathcal{F}_2[-1])$. □

3. Noncommutative tori, autoequivalences of $D^b(X)$ and related algebras

3.1. Morita autoequivalences of noncommutative tori and analogues of homogeneous coordinate rings

We refer to [15] for an introduction and a survey of main results in the theory of noncommutative tori. Recall that for every $\theta \in \mathbb{R}$ the algebra A_θ of smooth functions on the noncommutative torus T_θ is the algebra of series $\sum a_{n_1,n_2} U_1^{n_1} U_2^{n_2}$ in variables U_1, U_2 satisfying the relation

$$U_1 U_2 = \exp(2\pi \mathrm{i}\theta) U_2 U_1,$$

such that the coefficient function $(n_1, n_2) \to a_{n_1,n_2} \in \mathbb{C}$ is rapidly decreasing at infinity. By the definition, vector bundles on T_θ are finitely generated projective A_θ-modules (we always consider right modules). A complex structure on T_θ is given by a derivation $\delta_\tau : A_\theta \to A_\theta$, such that $\delta_\tau(U_1) = \tau$, $\delta_\tau(U_2) = 1$, where $\tau \in \mathbb{C} \setminus \mathbb{R}$ (following [14] we will usually impose the condition $\mathrm{Im}(\tau) < 0$). We denote by $T_{\theta,\tau}$ the noncommutative torus T_θ equipped with this complex structure. A holomorphic structure on a vector bundle is given by an operator $\nabla : E \to E$ on the corresponding projective right A_θ-module satisfying $\nabla(ea) = \nabla(e)a + e\delta_\tau(a)$, where $e \in E$, $a \in A_\theta$. As in [14] we only consider *standard* holomorphic vector bundles on $T_{\theta,\tau}$ that are given by certain family of standard holomorphic structures on *basic modules*.

Recall that a basic A_θ-module E is uniquely determined by its rank which is a primitive positive element in $\mathbb{Z}+\mathbb{Z}\theta$ (we assume that θ is irrational). The algebra of endomorphisms of E can be identified with $A_{\theta'}$ for some $\theta' \in \mathbb{R}$ and the functor $E' \mapsto E' \otimes_{A_{\theta'}} E$ is a Morita equivalence between the categories of right $A_{\theta'}$-modules and right A_θ-modules. It is known that θ' is necessarily of the form $\theta' = g\theta$ for some $g \in \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$. It is convenient to lift elements of $\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ to elements g of $\mathrm{SL}_2(\mathbb{Z})$ satisfying the condition $c\theta + d > 0$, where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For every such $g \in \mathrm{SL}_2(\mathbb{Z})$ there is a basic right A_θ-module $E_g(\theta)$ of rank $\mathrm{rk}(E_g(\theta)) = c\theta + d$ equipped with an isomorphism $\mathrm{End}_{A_\theta}(E_g(\theta)) \simeq A_{g\theta}$ (see [14, Section 1.1]).

Standard holomorphic structures on $E_g(\theta)$ are parametrized by the points of a complex elliptic curve $X_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. As we have shown in [14] (Proposition 4.1), fixing such a structure we obtain the Morita equivalence $E' \mapsto E' \otimes_{A_\theta} E_g(\theta)$ between the categories of holomorphic bundles on $T_{g\theta,\tau}$ and $T_{\theta,\tau}$ (preserving the subcategories of standard holomorphic bundles).

We are interested in the situation when $g(\theta) = \theta$ for some nontrivial $g \in \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$. It is easy to see that this happens exactly when either θ is rational or θ generates a real quadratic extension of $\mathbb{Q}$. In this case $E = E_g(\theta)$ has a natural structure of A_θ–A_θ-bimodule, so we can consider its tensor powers

$$E^{\otimes n} := E \otimes_{A_\theta} \cdots \otimes_{A_\theta} E(n\text{ times}) \simeq E_{g^n}(\theta),$$

where the last isomorphism follows from the general formula for the tensor product of basic modules (see, e.g. Proposition 1.2 of [14]). If we equip E with a standard holomorphic structure then $E^{\otimes n}$ acquires the induced holomorphic structure, so we can consider the corresponding space of holomorphic vectors $H^0(E^{\otimes n})$. Note that in order for these spaces to be nonzero we need to impose the condition $c > 0$ (where we assume that $\mathrm{Im}(\tau) < 0$; see [14, Proposition 2.5]). For $n = 0$ we set $E^{\otimes 0} = A_\theta$ and equip it with the standard holomorphic structure δ_τ. Now there is a natural structure of an associative algebra on

$$B_E := \bigoplus_{n \geq 0} H^0(E^{\otimes n})$$

given by the tensor product of holomorphic vectors. Clearly, this algebra is a direct generalization of the homogeneous coordinate ring. One can also define analogues of twisted homogeneous coordinate ring by changing the bimodule structure on E. Namely, one can leave the right A_θ-module structure the same and twist the left A_θ-module structure by some holomorphic (i.e., commuting with δ_τ) automorphism of A_θ.

Recall that in [14] we constructed an equivalence between the derived category of standard holomorphic vector bundles on $T_{\theta,\tau}$ and the full subcategory of stable objects in the derived category $D^b(X)$ of coherent sheaves on the elliptic curve $X = X_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. This equivalence sends each standard holomorphic bundle of rank $m\theta + n$ to a stable object in $D^b(X)$ of degree m and rank n. The image of the category of holomorphic bundles under this equivalence belongs to $\mathcal{C}^{-\theta^{-1}}$ (up to a shift). Moreover, the Morita autoequivalence $E' \mapsto E' \otimes_{A_\theta} E_g(\theta)$, where $g(\theta) = \theta$, corresponds to some autoequivalence $F : D^b(X) \to D^b(X)$ preserving $\mathcal{C}^{-\theta^{-1}} \subset D^b(X)$, such that $\pi(F) = g^t$ (the transposed matrix to g) Note that this is compatible with Proposition 2.6 since $g^t(-\theta^{-1}) = -\theta^{-1}$. It is also easy to see from the explicit formulas for the equivalence of [14] that F belongs to the subgroup $\mathrm{Aut}(D^b(X))^0 \subset \mathrm{Aut}(D^b(X))$ introduced in Section 2.1. By twisting the left A_θ-module structure on $E_g(\theta)$ with some holomorphic automorphisms of A_θ we can get more general autoequivalences F with $\pi(F) = g^t$.

Let $(\mathcal{F}_n, n \geq 0)$ be the image of the sequence of holomorphic bundles $(E^{\otimes n})$ under the above equivalence of categories. Then we have $\mathcal{F}_n = F^n(\mathcal{F}_0)$ and there is a natural isomorphism of algebras

$$B_E \simeq A_{F,\mathcal{F}_0}.$$

This isomorphism allows us to switch to the language of t-structures and autoequivalences of $D^b(X)$ in the further study of these algebras.

It is sometimes convenient to change the point of view slightly. Namely, the condition $g\theta = \theta$ is equivalent to the condition

$$r^2 - (a + d)r + 1 = 0,$$

where $r = c\theta + d$. In other words, r is an eigenvalue of g. Note that θ can be recovered from the pair (g, r) by the formula $\theta = (r - d)/c$. Thus, we can start with an arbitrary matrix g in $\mathrm{SL}_2(\mathbb{Z})$ having real positive eigenvalues and $c > 0$. Then fixing one of the eigenvalues r of g, we get a family of graded algebras

$$B_{g,r}(\tau) = \bigoplus_{n\geq 0} H^0(T_{\theta,\tau}, E^{\otimes n})$$

parametrized by $\tau \in \mathbb{C}$ such that $\mathrm{Im}(\tau) < 0$, where $\theta = (r - d)/c$, $E = E_g(\theta)$ is equipped with a standard holomorphic structure $\bar{\nabla}_0$ (see [14]). Note that for every upper-triangular matrix $u \in \mathrm{SL}_2(\mathbb{Z})$ one has $B_{ugu^{-1},r}(\tau) \simeq B_{g,r}(\tau)$, so this family of algebras really lives on a double covering of $\mathrm{SL}_2(\mathbb{Z})/\mathrm{Ad}(\mathbb{Z})$, where $\mathbb{Z}$ is embedded as upper-triangular matrices in $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 3.1. *There is an isomorphism of graded algebras*

$$B_{g,r}(\tau)^{\mathrm{opp}} \simeq B_{g',r^{-1}}(\tau),$$

where

$$g' = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Proof. We have a natural isomorphism $A_{-\theta} \simeq A_\theta^{\mathrm{opp}}$, identical on generators U_1 and U_2. Thus, we can consider $E = E_g(\theta)$ as an $A_{-\theta}$–$A_{-\theta}$-bimodule. It is easy to see that as such a bimodule E is isomorphic to $E_{g'}(-\theta)$. Moreover, this isomorphism is compatible with standard holomorphic structures up to a scalar. This implies the result. □

3.2. Algebras associated with autoequivalences and t-structures

We want to look at the algebras $A_{F,\mathcal{F}}$, where $\mathcal{F} \in D^b(X)$ is a stable object, $F : D^b(X) \to D^b(X)$ is an autoequivalence. In order to get interesting algebras we would like to impose the condition $F^n(\mathcal{F}) \not\simeq \mathcal{F}$ and $\mathrm{Hom}(\mathcal{F}, F^n(\mathcal{F})) \neq 0$ for all sufficiently large $n \geq 0$. We are going to show that this condition implies that the corresponding element $\pi(F) \in \mathrm{SL}_2(\mathbb{Z})$ has positive real eigenvalues.

Lemma 3.2. *For $g \in \mathrm{SL}_2(\mathbb{Z})$ and $v \in \mathbb{Z}^2 \setminus \{0\}$ the following conditions are equivalent*:

(i) $\chi(v, g^n v) > 0$ *for all sufficiently large n*;
(i)′ $\chi(v, g^n v) > 0$ *for all $n > 0$*;
(ii) *g has positive real eigenvalues and* $\chi(v, gv) > 0$.

Proof. First, let us show that (i) implies (ii). Set $N = \mathrm{tr}(g)$. Then $g^2 - Ng + 1 = 0$. Hence,

$$\chi(v, g^n v) - N\chi(v, g^{n-1}v) + \chi(v, g^{n-2}v) = 0.$$

From this we immediately derive that

$$\sum_{n\geq 0} \chi(v, g^n v)t^n = \frac{Mt}{1 - Nt + t^2},$$

where $M = \chi(v, gv)$. Therefore, condition (ii) implies that all coefficients of the series $M(1 - Nt + t^2)^{-1}$ except for a finite number are positive. It is easy to see that for $N \geq 2$ (i.e., when both roots of the equation $t^2 - Nt + 1 = 0$ are real and positive) all coefficients of the series $(1 - Nt + t^2)^{-1}$ are positive. By the change of variables $t \mapsto -t$ this implies that the series $(1 - Nt + t^2)^{-1}$ is alternating for $N \leq -2$. It is also easy to see directly that for $N = \pm 1$ or $N = 0$ this series still has infinitely many negative coefficients. Hence, (i) implies that $M > 0$ and $N \geq 2$. The same argument shows that (ii) implies (i)′. □

Proposition 3.3. *Let $F : D^b(X) \to D^b(X)$ be an autoequivalence with $\pi(F) = g \in \mathrm{SL}_2(\mathbb{Z})$, $\mathcal{F}$ be a stable object of $D^b(X)$. Then the following conditions are equivalent*:

(i) $F^n(\mathcal{F}) \not\simeq \mathcal{F}$ *for $n > 0$ and* $\mathrm{Hom}(\mathcal{F}, F^n(\mathcal{F})) \neq 0$ *for all sufficiently large n*;
(ii) *g has positive real eigenvalues, $M = \chi(\mathcal{F}, F(\mathcal{F})) > 0$, and there exists $\theta \in \mathbb{R} \cup \{\infty\}$ such that F preserves $\mathcal{C}^\theta$.*

Under these conditions the Hilbert series of $A_{F,\mathcal{F}}$ is equal to

$$H_{A_{F,\mathcal{O}_X}}(t) = 1 + \frac{Mt}{1 - Nt + t^2},$$

where $N = \mathrm{tr}(g)$. Also, if $(\theta, 1) \in \mathbb{R}^2$ is an eigenvector of g then F preserves the subcategory $\mathcal{C}^\theta \subset D^b(X)$ (if $(1, 0)$ is an eigenvector then we set $\theta = \infty$).

Proof. (i) ⇒ (ii). Note that for every pair of nonisomorphic stable objects $\mathcal{G}, \mathcal{G}' \in D^b(X)$ the graded space $\mathrm{Hom}^\bullet(\mathcal{G}, \mathcal{G}')$ is always concentrated in one degree. Hence, (i) implies that $\chi(\mathcal{F}, F^n(\mathcal{F})) = \chi(v_0, g^n v_0) > 0$ for all sufficiently large n, where $v_0 = v_{\mathcal{F}} \in \mathbb{Z}^2$. By Lemma 3.2 this implies that g has positive eigenvalues and that $M > 0$. Also, from the proof of this lemma we obtain the formula for the Hilbert series of $A_{F,\mathcal{F}}$. Now let $u \in \mathbb{R}^2$ be an eigenvector of g. Rescaling u we can assume that either $u = (\theta, 1)$ or $u = (1, 0)$. In the latter case we set $\theta = \infty$. In either case we have $g(\theta) = \theta$ under the fractional-linear action. Therefore, by Proposition 2.6 we have $F(\mathcal{C}^\theta) = \mathcal{C}^\theta[m]$ for some $m \in \mathbb{Z}$. Since $\mathcal{F}$ belongs to $\mathcal{C}^\theta[i]$ for some $i \in \mathbb{Z}$, the nonvanishing of $\mathrm{Hom}(\mathcal{F}, F^n(\mathcal{F}))$ for $n \gg 0$ implies that $m = 0$.

(ii) ⇒ (i). Let us consider vectors

$$v_n = v_{F^n(\mathcal{F})} = g^n v_0 \in \mathbb{Z}^2 \subset \mathbb{R}^2.$$

Since $\chi(v_0, v_1) = M > 0$, it follows that $\chi(v_n, v_{n+1}) > 0$ for all $n \geq 0$. Let m be an integer such that $\mathcal{F} \in \mathcal{C}^\theta[m]$. Then all the vectors v_n belong to the closed half-plane $(-1)^m \overline{H_\theta} \subset \mathbb{R}^2$ (see Proposition 2.5). Hence, $\chi(v_i, v_j) \geq 0$ for $i < j$. Moreover, for a pair of stable objects $\mathcal{G}, \mathcal{G}' \in \mathcal{C}^\theta$ the vectors $v_{\mathcal{G}}$ and $v_{\mathcal{G}'}$ can be proportional only if $v_{\mathcal{G}} = v_{\mathcal{G}'}$. Hence, we have $\chi(v_i, v_j) > 0$ for $i < j$. It remains to apply Lemma 2.7. □

Remark 3.4.

1. It is easy to deduce from the proof that for a pair $(F, \mathcal{F})$ such that $g = \pi(F) \in \mathrm{SL}_2(\mathbb{Z})$ has positive real eigenvalues and $M > 0$, the conditions of the above proposition will be satisfied for $(F[n], \mathcal{F})$ for some $n \in 2\mathbb{Z}$.
2. If F satisfies the equivalent conditions of the above proposition, g has two distinct eigenvalues and $(\theta_1, 1)$, $(\theta_2, 1)$ are the corresponding eigenvectors, then F preserves both subcategories $\mathcal{C}^{\theta_1}$ and $\mathcal{C}^{\theta_2}$. Hence, F also preserves $\mathcal{C}^{\theta_1} \cap \mathcal{C}^{\theta_2}$. Moreover, it is easy to see that $\mathcal{C}^{\theta_1} \cap \mathcal{C}^{\theta_2}$ is a "half" of the natural F-invariant torsion theory in each of the categories $\mathcal{C}^{\theta_1}$, $\mathcal{C}^{\theta_2}$ and that these categories are tiltings of each other with respect to these torsion theories.

One can rewrite the Hilbert series of $A_{F,\mathcal{F}}$ as follows:

$$H_{A_{F,\mathcal{F}}}(t) = \frac{1 + (M - N)t + t^2}{1 - Nt + t^2}.$$

In particular, we notice that the series $H_{A_{F,\mathcal{F}}}(-t)^{-1}$ has similar form but with N and $M - N$ switched. Recall that if a graded algebra A is Koszul then one has $H_{A^!}(t) = H_A(-t)^{-1}$, where $A^!$ is the Koszul dual algebra. Below we will show that under appropriate conditions the algebra $A_{F,\mathcal{F}}$ is Koszul with the dual also of the form $A_{F',\mathcal{F}}$.

3.3. *Koszul duality*

For every stable object $\mathcal{F} \in D^b(X)$ we denote by $R_{\mathcal{F}}$ the right twist corresponding to $\mathcal{F}$. This is an autoequivalence on $D^b(X)$ such that one has exact triangles

$$\mathcal{G} \to \mathrm{Hom}^\bullet(\mathcal{G}, \mathcal{F})^* \otimes \mathcal{F} \to R_{\mathcal{F}}\mathcal{G} \to \mathcal{G}[1]$$

for all $\mathcal{G} \in D^b(X)$ (see [18]; our notation is slightly different). The quasi-inverse autoequivalence is the left twist $L_{\mathcal{F}}$, such that one has an exact triangle

$$L_{\mathcal{F}}(\mathcal{G}) \to \mathrm{Hom}^{\bullet}(\mathcal{F}, \mathcal{G}) \otimes \mathcal{F} \to \mathcal{G} \to L_{\mathcal{F}}(\mathcal{G})[1].$$

Theorem 3.5. *Let $(F, \mathcal{F})$ be a pair satisfying the equivalent conditions of Proposition 3.3. Let $\pi(F) = g \in \mathrm{SL}_2(\mathbb{Z})$, $N = \mathrm{tr}(g)$, $M = \chi(\mathcal{F}, F(\mathcal{F}))$, and let $A = A_{F,\mathcal{F}}$ be the corresponding graded algebra. Then*

(a) *A is generated by A_1 over $A_0 = k$ if and only if $M \geq N + 1$, or $M = N$ and*

$$\det F^2(\mathcal{F}) \not\simeq (\det F(\mathcal{F}))^N \otimes \det(\mathcal{F})^{-1};$$

(b) *A is a quadratic algebra if and only if $M \geq N + 2$, or $M = N + 1$ and*

$$\det F^3(\mathcal{F}) \not\simeq (\det F^2(\mathcal{F}))^{N+1} \otimes (\det F(\mathcal{F}))^{-N-1} \otimes \det(\mathcal{F});$$

(c) *A is Koszul if and only if $M \geq N + 2$. Moreover, in this case one has the following isomorphism for the quadratic dual algebra*:

$$A^! \simeq A_{R_{\mathcal{F}} \circ F^{-1}, \mathcal{F}}.$$

Let (r, r^{-1}) be the eigenvalues of g and let (u, u') be the corresponding eigenvectors, so that $gu = ru$, $gu' = r^{-1}u'$. Let $u^* : \mathbb{R}^2 \to \mathbb{R}$ be the functional defined by $u^*(u) = 1$, $u^*(u') = 0$. We can choose u in such a way that $u^*(v_0) > 0$, where $v_0 = v_{\mathcal{F}}$ (note that v_0 cannot be an eigenvector of g since $M = \chi(v_0, gv_0) > 0$). Consider the half-plane $H = \{v \in \mathbb{R}^2 | u^*(v) > 0\}$. Let $\mathcal{C} = \mathcal{C}^{\theta}[m]$ for appropriate $\theta \in \mathbb{R} \cup \{\infty\}$ and $m \in \mathbb{Z}$, so that $\mathcal{F} \in \mathcal{C}$ and the vectors $(v_{\mathcal{G}}, \mathcal{G} \in \mathcal{C})$ belong to the closure of H. Then F preserves $\mathcal{C}$ (see Proposition 3.3). For an object $\mathcal{G}$ of $D^b(X)$ we set

$$\mathrm{rk}_{\mathcal{C}}(\mathcal{G}) = \frac{u^*(v_{\mathcal{G}})}{u^*(v_0)},$$

so that $\mathrm{rk}_{\mathcal{C}}(\mathcal{G}) \geq 0$ for all $\mathcal{G} \in \mathcal{C}$. From the definition of u^* we immediately derive that

$$\mathrm{rk}_{\mathcal{C}}(F(\mathcal{G})) = r \cdot \mathrm{rk}_{\mathcal{C}}(\mathcal{G}).$$

Note that $\mathcal{C}$ contains the subcategory equivalent to the category of stable bundles on a noncommutative 2-torus and $\mathrm{rk}_{\mathcal{C}}$ is proportional to the rank function on such bundles. Let us denote $\mathcal{F}_n = F^n(\mathcal{F}) \in \mathcal{C}$. Since $\mathrm{rk}_{\mathcal{C}}(\mathcal{F}) = 1$, we obtain that

$$\mathrm{rk}_{\mathcal{C}}(\mathcal{F}_n) = r^n.$$

To prove the above theorem we are going to use the twist functors $R_{\mathcal{F}_n} : D^b(X) \to D^b(X)$. More precisely, let us consider the objects

$$\mathcal{F}'_n := R_{\mathcal{F}_n} R_{\mathcal{F}_{n-1}} \cdots R_{\mathcal{F}_1}(\mathcal{F}_0) \in D^b(X),$$

where $n > 0$. It is convenient to extend this definition to $n = 0$ by setting $\mathcal{F}'_0 = \mathcal{F}_0$. As we will see, the properties of the algebra A depend on whether some (or all) $\mathcal{F}'_n$ belong to the subcategory $\mathcal{C}$ and also on the vanishing of some (or all) spaces $\mathrm{Hom}^1(\mathcal{F}'_n, \mathcal{F}_m)$ for $n < m$.

Lemma 3.6. *Consider the following generating series*

$$F(t,u) = \sum_{n\geq 0, k\geq 0} \chi(\mathcal{F}'_n, \mathcal{F}_{n+k+1}) t^n u^k, \qquad R(t) = \sum_{n\geq 0} \mathrm{rk}_{\mathcal{C}}(\mathcal{F}'_n) t^n.$$

Then one has

$$F(t,u) = \frac{M(1+tu)}{(1-Nu+u^2)(1-(M-N)t+t^2)}, \qquad R(t) = \frac{1+r^2t}{1-(M-N)rt+r^2t^2}.$$

Proof. By the definition of $\mathcal{F}'_n$ we have an exact triangle

$$\mathcal{F}'_{n-1} \to \mathrm{Hom}^\bullet(\mathcal{F}'_{n-1}, \mathcal{F}_n)^* \otimes \mathcal{F}_n \to \mathcal{F}'_n \to \mathcal{F}'_{n-1}[1]. \tag{3.1}$$

This implies the following relations:

$$\chi(\mathcal{F}'_n, \mathcal{F}_m) = \chi(\mathcal{F}'_{n-1}, \mathcal{F}_n)\chi(\mathcal{F}_n, \mathcal{F}_m) - \chi(\mathcal{F}'_{n-1}, \mathcal{F}_m), \quad m > n \geq 1,$$
$$\mathrm{rk}_{\mathcal{C}}(\mathcal{F}'_n) = \chi(\mathcal{F}'_{n-1}, \mathcal{F}_n) r^n - \mathrm{rk}_{\mathcal{C}}(\mathcal{F}'_{n-1}), \quad n \geq 1.$$

Note that $\chi(\mathcal{F}_n, \mathcal{F}_m) = \chi(\mathcal{F}_0, \mathcal{F}_{m-n})$ for $m > n$ is a coefficient of the Hilbert series of A:

$$H(t) = 1 + \sum_{n\geq 1} \chi(\mathcal{F}_0, \mathcal{F}_n) t^n.$$

Therefore, denoting

$$F_0(t) = F(t,0) = \sum_{n\geq 0} \chi(\mathcal{F}'_n, \mathcal{F}_{n+1}) t^n,$$

we obtain the equations

$$(u+t)F(t,u) = H(u)(1+tF_0(t)) - 1, \qquad R(t) = 1 + rtF_0(rt) - tR(t).$$

Substituting $u = -t$ into the first equation we get

$$F_0(t) = \frac{H(-t)^{-1} - 1}{t},$$

and therefore,

$$F(t,u) = \frac{H(u)H(-t)^{-1} - 1}{u+t}, \qquad R(t) = \frac{H(-rt)^{-1}}{1+t}.$$

It remains to use the formula

$$H(t) = \frac{1+(M-N)t+t^2}{1-Nt+t^2}$$

that was proven in Proposition 3.3. □

Proof of Theorem 3.5.

(a) If the map $A_1 \otimes A_1 \to A_2$ is surjective then $M^2 = (\dim A_1)^2 \geq \dim A_2 = MN$, hence $M \geq N$.

Conversely, assume that $M \geq N$. Then we claim that $\mathcal{F}'_1 \in \mathcal{C}$ and $\mathrm{Hom}^1(\mathcal{F}'_1, \mathcal{F}_m) = 0$ for $m > 2$. Indeed, since $\mathcal{F}'_1 = R_{\mathcal{F}_1}(\mathcal{F}_0)$, it is a stable object of $D^b(X)$. Hence, the exact triangle (3.1) for $n = 1$ implies that either $\mathcal{F}'_1 \in \mathcal{C}$ or $\mathcal{F}'_1 \in \mathcal{C}[1]$. To check that $\mathcal{F}'_1 \in \mathcal{C}$ it suffices to prove the inequality $\mathrm{rk}_{\mathcal{C}}(\mathcal{F}'_1) > 0$. But

$$\mathrm{rk}_{\mathcal{C}}(\mathcal{F}'_1) = Mr - 1 = (M - N)r + r^2 \geq r^2 > 0,$$

which proves our first claim (we used the equality $N = r + r^{-1}$). On the other hand, since $\mathcal{F}'_1$ and $\mathcal{F}_m$ are both stable objects of $\mathcal{C}$, the vanishing of $\mathrm{Hom}^1(\mathcal{F}'_1, \mathcal{F}_m)$ would follow from the inequality $\chi(\mathcal{F}'_1, \mathcal{F}_m) > 0$ (see Lemma 2.7). From Lemma 3.6 we get

$$\sum_{k \geq 0} \chi(\mathcal{F}'_1, \mathcal{F}_{k+2}) u^k = \frac{\partial F}{\partial t}(0, u) = \frac{M(M - N + u)}{1 - Nu + u^2}.$$

The latter series has positive coefficients except maybe for the constant term. Hence, $\mathrm{Hom}^1(\mathcal{F}'_1, \mathcal{F}_m) = 0$ for $m > 2$. Now from the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}^0(\mathcal{F}'_1, \mathcal{F}_m) \to \mathrm{Hom}^0(\mathcal{F}_0, \mathcal{F}_1) \otimes \mathrm{Hom}^0(\mathcal{F}_1, \mathcal{F}_m) \\ &\to \mathrm{Hom}^0(\mathcal{F}_0, \mathcal{F}_m) \to \mathrm{Hom}^1(\mathcal{F}'_1, \mathcal{F}_m), \end{aligned} \tag{3.2}$$

we derive the surjectivity of the map $A_1 \otimes A_{m-1} \to A_m$ for $m > 2$. In the case $M > N$ the above argument shows that $\mathrm{Hom}^1(\mathcal{F}'_1, \mathcal{F}_2) = 0$, hence in this case the map $A_1 \otimes A_1 \to A_2$ is also surjective. On the other hand, if $M = N$ then $v_{\mathcal{F}'_1} = v_{\mathcal{F}_2}$, so either $\mathrm{Hom}^\bullet(\mathcal{F}'_1, \mathcal{F}_2) = 0$, or $\mathcal{F}'_1 \simeq \mathcal{F}_2$. Hence, in this case the map $A_1 \otimes A_1 \to A_2$ is surjective if and only if $\det(\mathcal{F}'_1) \not\simeq \det(\mathcal{F}_2)$. Using the triangle (3.1) we get that $\det(\mathcal{F}'_1) \simeq \det(\mathcal{F}_1)^N \otimes \det(\mathcal{F}_0)^{-1}$ which leads to the condition in the formulation of part (a).

(b) By the result of part (a) we can assume that A is generated by A_1. The statement that the algebra A is quadratic is equivalent to surjectivity of the natural maps

$$A_{m-2} \otimes I \to \ker(A_{m-1} \otimes A_1 \to A_m) \tag{3.3}$$

for all $m \geq 3$, where $I = \ker(A_1 \otimes A_1 \to A_2)$ is the space of quadratic relations. From the exact sequences (3.2) above we see that I can be identified with $\mathrm{Hom}^0(\mathcal{F}'_1, \mathcal{F}_2)$ and that the map (3.3) can be identified with the natural map

$$\mathrm{Hom}^0(\mathcal{F}_2, \mathcal{F}_m) \otimes \mathrm{Hom}^0(\mathcal{F}'_1, \mathcal{F}_2) \to \mathrm{Hom}^0(\mathcal{F}'_1, \mathcal{F}_m), \quad m \geq 3.$$

Now the exact triangle (3.1) for $n = 2$ shows that the kernel and the cokernel of this map are $\mathrm{Hom}^0(\mathcal{F}'_2, \mathcal{F}_m)$ and $\mathrm{Hom}^1(\mathcal{F}'_2, \mathcal{F}_m)$, respectively (we use the facts about $\mathcal{F}'_1$ proven in part (a)). Therefore, to prove that the algebra A is quadratic it suffices to show that $\mathrm{Hom}^1(\mathcal{F}'_2, \mathcal{F}_m) = 0$ for $m \geq 3$.

The same argument as in (a) shows that either $\mathcal{F}'_2 \in \mathcal{C}$ or $\mathcal{F}'_2 \in \mathcal{C}[1]$. Moreover, we have

$$\mathrm{rk}_{\mathcal{C}}(\mathcal{F}'_2) = ((M - N)^2 - 1)r^2 + (M - N)r^3,$$

so for $M \geq N+1$ we have $\mathrm{rk}_{\mathcal{C}}(\mathcal{F}'_2) > 0$, hence $\mathcal{F}'_2 \in \mathcal{C}$. On the other hand, using Lemma 3.6 we get

$$\sum_{k\geq 0} \chi(\mathcal{F}'_2, \mathcal{F}_{k+3})u^k = \frac{(M-N)^2 - 1 + (M-N)u}{1 - Nu + u^2}.$$

Thus, if $M - N \geq 2$ then $\chi(\mathcal{F}'_2, \mathcal{F}_m) > 0$ for all $m \geq 3$. Since $\mathcal{F}'_2$ is a stable object of $\mathcal{C}$, this implies the required vanishing of $\mathrm{Hom}^1(\mathcal{F}'_2, \mathcal{F}_m)$. If $M = N+1$ then $\chi(\mathcal{F}'_2, \mathcal{F}_m) > 0$ for $m \geq 4$ while $\chi(\mathcal{F}'_2, \mathcal{F}_3) = 0$. Hence, in this case $\mathrm{Hom}^1(\mathcal{F}'_2, \mathcal{F}_m) = 0$ for $m \geq 4$ and either $\mathrm{Hom}^\bullet(\mathcal{F}'_2, \mathcal{F}_3) = 0$ or $\mathcal{F}'_2 \simeq \mathcal{F}_3$. The latter isomorphism occurs exactly when $\det(\mathcal{F}'_2) \simeq \det(\mathcal{F}_3)$. It remains to use (3.1) to get an isomorphism

$$\det(\mathcal{F}'_2) \simeq \det(\mathcal{F}_2)^{N+1} \otimes \det(\mathcal{F}'_1)^{-1} \simeq \det(\mathcal{F}_2)^{N+1} \otimes \det(\mathcal{F}_1)^{-N-1} \otimes \det(\mathcal{F}_0).$$

Conversely, if the algebra A is quadratic then

$$\chi(\mathcal{F}'_2, \mathcal{F}_3) = M\chi(\mathcal{F}'_1, \mathcal{F}_2) - \chi(\mathcal{F}'_1, \mathcal{F}_3) \geq 0,$$

hence $(M-N)^2 \geq 1$. Since we already know that $M - N \geq 0$ this implies that $M - N \geq 1$.

(c) If the algebra A is Koszul then its Hilbert series $H(t)$ has the property that $H(-t)^{-1}$ has nonnegative coefficients. But

$$H(-t)^{-1} = \frac{1 + Nt + t^2}{1 - (M-N)t + t^2} = 1 + \frac{Mt}{1 - (M-N)t + t^2},$$

so this series has nonnegative coefficients only if $M - N \geq 2$.

Conversely, assume that $M - N \geq 2$. We claim that all the objects $\mathcal{F}'_n$ belong to $\mathcal{C}$ and that $\mathrm{Hom}^1(\mathcal{F}'_n, \mathcal{F}_m) = 0$ for $m > n$. We argue by induction. Assume that the assertion is true for $n-1$. Looking at the exact triangle (3.1) we conclude as in part (a) that either $\mathcal{F}'_n \in \mathcal{C}$ or $\mathcal{F}'_n \in \mathcal{C}[1]$. Since by Lemma 3.6 we also have $\mathrm{rk}_{\mathcal{C}}(\mathcal{F}'_n) > 0$ this implies that $\mathcal{F}'_n \in \mathcal{C}$. On the other hand, the same Lemma shows that $\chi(\mathcal{F}'_n, \mathcal{F}_m) > 0$ for $m > n$. By Lemma 2.7 this implies the vanishing of $\mathrm{Hom}^1(\mathcal{F}'_n, \mathcal{F}_m)$ for $m > n$.

Now let us set

$$K_n := \bigoplus_{m>n} \mathrm{Hom}^0(\mathcal{F}'_n, \mathcal{F}_m).$$

Then K_n has a natural structure of (graded) left A-module and from (3.1) we deduce the following exact sequences of A-modules:

$$0 \to K_n \to A(-n) \otimes \mathrm{Hom}(\mathcal{F}'_{n-1}, \mathcal{F}_n) \to K_{n-1} \to 0, \quad n \geq 1,$$

where $A(-n)$ is the free A-module with one generator in degree n. Since K_0 is the augmentation ideal $A_+ \subset A$, putting these sequences together we obtain a free resolution of the trivial module k of the form

$$\cdots \to A(-n) \otimes \mathrm{Hom}(\mathcal{F}'_{n-1}, \mathcal{F}_n) \to \cdots \to A(-1) \otimes \mathrm{Hom}(\mathcal{F}'_0, \mathcal{F}_1) \to A.$$

This implies that A is Koszul and $A^!_n \simeq \mathrm{Hom}(\mathcal{F}'_{n-1}, \mathcal{F}_n)^*$ for $n \geq 1$.

To prove the last statement of the theorem we observe that the exact triangle (3.1) shows that

$$A_n^! \simeq \mathrm{Hom}(\mathcal{F}'_{n-1}, \mathcal{F}_n)^* \simeq \mathrm{Hom}(\mathcal{F}_n, \mathcal{F}'_n).$$

Since $R_{\mathcal{F}_n} \simeq F^n R_{\mathcal{F}} F^{-n}$, we also have

$$\mathcal{F}'_n \simeq F^n (R_{\mathcal{F}} F^{-1})^n (\mathcal{F}).$$

Therefore,

$$A_n^! \simeq \mathrm{Hom}(\mathcal{F}_n, \mathcal{F}'_n) \simeq \mathrm{Hom}(\mathcal{F}, (R_{\mathcal{F}} F^{-1})^n (\mathcal{F})).$$

It is easy to check that this isomorphism is compatible with the multiplication on $A^!$ and on $A_{R_{\mathcal{F}} F^{-1}, \mathcal{F}}$. □

Example 3.7.

1. If $\alpha_F = 1$ then by Proposition 2.3 part (b) of the above theorem states that $A_{F,\mathcal{F}}$ is quadratic iff $M \geq N + 2$. For example, this is the case for $F = (\otimes L)$, where L is a line bundle. This leads to the well-known statement that the corresponding algebra $A_{F,\mathcal{O}_X} = \oplus_{n\geq 0} H^0(X, L^n)$ is quadratic iff $\deg(L) \geq 4$ (then it is also Koszul). More generally, if $F = (\otimes L) \circ \sigma^*$, where σ is an automorphism of X, then $A_{F,\mathcal{O}_X}$ is the so-called twisted coordinate algebra attached to the pair (L, σ). Such algebras were considered in [1]. For example, if σ is a translation then such an algebra is quadratic iff $\deg(L) \geq 4$, in which case it is also Koszul.
2. Consider the case $M = N + 1$, $\alpha_F = -1$. Then the algebra A is often quadratic but never Koszul. The quadratic dual has $A_3^! = 0$. For example, if L is a line bundle of degree 3 such that $[-1]^* L \otimes L^{-1}$ is not of order 2, then these conditions are satisfied for $F = (\otimes L) \circ [-1]^*$. The corresponding algebra is
$$A = k \oplus H^0(L) \oplus H^0(L \otimes [-1]^* L) \oplus H^0(L \otimes [-1]^* L \otimes L) \oplus \cdots$$
with the multiplication rule $f * g = f[(-1)^{\deg(f)}]^* g$.
3. If $M = N + 1$ and $\alpha_F = 1$ then the algebra A is not quadratic: one has to add one cubic relation to the quadratic relations.

The following result allows to check under what conditions Theorem 3.5 can be applied to sufficiently high powers of a given autoequivalence.

Proposition 3.8. *Assume that an element $g \in \mathrm{SL}_2(\mathbb{Z})$ has positive real eigenvalues and that the vector $v \in \mathbb{Z}^2 \setminus \{0\}$ satisfies $M := \chi(v, gv) > 0$. Then the following conditions are equivalent*:

(i) $\chi(v, g^n v) - \mathrm{tr}(g^n) \to +\infty$ *as* $n \to +\infty$;
(ii) $\chi(v, g^n v) - \mathrm{tr}(g^n) \geq 0$ *for some* $n > 0$;
(iii) $M > r_1 - r_2$, *where* $r_1 \geq r_2$ *are eigenvalues of* g;
(iv) *either* g *is unipotent, or* $M \geq N$, *where* $N := \mathrm{tr}(g)$.

Proof. Note that by Lemma 3.2 we have $\chi(v, g^n v) > 0$ for all $n > 0$. Moreover, from the proof of that Lemma we get

$$\sum_{n\geq 1} \chi(v, g^n v)t^n = \frac{Mt}{1 - Nt + t^2}.$$

If g is unipotent then we have $N = \mathrm{tr}(g^n) = 2$, while $\chi(v, g^n v) = nM$, so the assertion is clear.

Now assume that g has two distinct eigenvalues $r > r^{-1} > 0$. Then from the above formula we get

$$\chi(v, g^n v) = M\frac{r^n - r^{-n}}{r - r^{-1}}.$$

On the other hand, $\mathrm{tr}(g^n) = r^n + r^{-n}$. Using these formulas it is not difficult to show the equivalence of (i)–(iv). Indeed, clearly, (i) implies (ii). The implication (ii) ⇒ (iii) follows immediately from the chain of inequalities

$$\frac{Mr^n}{r - r^{-1}} > \frac{M(r^n - r^{-n})}{r - r^{-1}} \geq r^n + r^{-n} > r^n.$$

To prove (iii) ⇒ (iv) we note that $r - r^{-1} = \sqrt{N^2 - 4}$. Thus, the inequality $M > r - r^{-1}$ implies that $M^2 > N^2 - 4$, hence $M \geq N$ (since $N \geq 3$ in our case). Finally, if $M \geq N$ then $M > r - r^{-1}$, in which case $\chi(v, g^n v) - \mathrm{tr}(g^n) \to +\infty$ as $n \to +\infty$. This proves (iv) ⇒ (i). □

4. Ampleness and noncommutative Proj

4.1. Ampleness criterion

Let A be a graded k-algebra of the form $A = \oplus_{i\geq 0} A_i$, where $A_0 = k$. Recall that a finitely generated right graded A-module M is called *coherent* if for every finite collection of homogeneous elements $m_1, \ldots, m_n \in M$ the (right) A-module of relations between $m_1, \ldots, m_n$ is finitely generated. Coherent modules form an Abelian subcategory in the category of all modules. An algebra A is called right coherent if it is finitely generated and is coherent as a right module over itself. We denote by cohproj A the quotient of the category of coherent A-modules by the Serre subcategory of bounded coherent modules. Below we are going to show that the categories $\mathcal{C}^\theta$, where θ is a quadratic irrationality (or a rational number), are equivalent to such quotient categories for appropriate algebras of the form $A_{F,\mathcal{F}}$.

Let $\mathcal{C}$ be an Abelian category equipped with an autoequivalence $F : \mathcal{C} \to \mathcal{C}$. For simplicity we will assume that F is an automorphism of $\mathcal{C}$ (the general case can be reduced to this one, see [2]). Under appropriate ampleness assumptions the category $\mathcal{C}$ can be recovered from the algebra $A_{F,\mathcal{O}}$, where $\mathcal{O}$ is an object of $\mathcal{C}$. Recall (see [13,19]) that a sequence $(\mathcal{O}_n, n \in \mathbb{Z})$ of objects of $\mathcal{C}$ is called *ample* if the following two conditions hold: (i) for every surjection $X \to Y$ in $\mathcal{C}$ the induced map $\mathrm{Hom}_{\mathcal{C}}(\mathcal{O}_n, X) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}_n, Y)$ is surjective for all $n \ll 0$;

(ii) for every object $X \in \mathcal{C}$ and every $n \in \mathbb{Z}$ there exists a surjection $\oplus_{j=1}^{s} O_{i_j} \to X$, where $i_j < n$ for all j. The main theorem of [13] implies that if the sequence $(F^n O, n \in \mathbb{Z})$ is ample then the algebra $A_{F,O}$ is right coherent and the categories $\mathcal{C}$ and cohproj $A_{F,O}$ are equivalent. Similar result for Noetherian categories was proven by Artin and Zhang [2]. The following proposition shows that we do need a more general theorem of [13], since for irrational θ the categories $\mathcal{C}^\theta$ are not Noetherian.

Proposition 4.1. *Assume that θ is irrational. Then every nonzero object in $\mathcal{C}^\theta$ is not Noetherian.*

Proof. It suffices to prove that every stable object $\mathcal{F} \in \mathcal{C}^\theta$ is not Noetherian. Recall that the vector $v_{\mathcal{F}} = (\deg(\mathcal{F}), \mathrm{rk}(\mathcal{F})) \in \mathbb{Z}^2$ satisfies $\deg(\mathcal{F}) - \mathrm{rk}(\mathcal{F})\theta > 0$. Moreover, since $\mathcal{F}$ is stable, the numbers $\deg(\mathcal{F})$ and $\mathrm{rk}\,\mathcal{F}$ are relatively prime. Let (m, n) be the unique pair of integers such that $m\,\mathrm{rk}(\mathcal{F}) - n\deg(\mathcal{F}) = 1$ and $0 < m - n\theta < \deg(\mathcal{F}) - \mathrm{rk}(\mathcal{F})$. There exists a stable object $\mathcal{F}' \in \mathcal{C}^\theta$ with $v_{\mathcal{F}'} = (m, n)$ (see Proposition 2.5). By Lemma 2.7 one has $\dim \mathrm{Hom}(\mathcal{F}, \mathcal{F}') = 1$ and $\mathrm{Hom}^1(\mathcal{F}, \mathcal{F}') = 0$. This implies that there is an exact triangle

$$\mathcal{F}'[-1] \to L_{\mathcal{F}}(\mathcal{F}') \to \mathcal{F} \to \mathcal{F}' \to \cdots,$$

where $L_{\mathcal{F}}$ is the left twist functor corresponding to $\mathcal{F}$. Note that the object $L_{\mathcal{F}}(\mathcal{F}')$ is stable, so either $L_{\mathcal{F}}(\mathcal{F}') \in \mathcal{C}^\theta$, or $L_{\mathcal{F}}(\mathcal{F}') \in \mathcal{C}^\theta[-1]$. But the vector $v_{L_{\mathcal{F}}(\mathcal{F}')} = v_{\mathcal{F}} - v'_{\mathcal{F}}$ lies in the half-plane $\{(x, y)|x - y\theta > 0\}$, hence $L_{\mathcal{F}}(\mathcal{F}')$ is in $\mathcal{C}^\theta$. This means that $\mathcal{F}'$ is a proper quotient-object of $\mathcal{F}$ in $\mathcal{C}^\theta$. Iterating this procedure we will obtain an infinite sequence $\mathcal{F} \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots$, where $\mathcal{F}_{n+1}$ is a proper quotient of $\mathcal{F}_n$. $\square$

The following theorem gives a criterion of ampleness for sequences of the form $(F^n\mathcal{F}, n \in \mathbb{Z})$ in the categories $\mathcal{C}^\theta$.

Theorem 4.2. *Let $F : D^b(X) \to D^b(X)$ be an autoequivalence such that the element $g = \pi(F) \in \mathrm{SL}_2(\mathbb{Z})$ has distinct positive real eigenvalues. Let $u = (x, y) \in \mathbb{R}^2$ be an eigenvector of g with the eigenvalue < 1 and let $\theta = x/y$ (if $y = 0$ then $\theta = \infty$). Let $\mathcal{F}_0$ be a stable object of $\mathcal{C}^\theta$ and let $v_0 = (\deg \mathcal{F}_0, \mathrm{rk}\,\mathcal{F}_0)$ be the corresponding primitive vector in $\mathbb{Z}^2$. Assume that $F(\mathcal{F}_0) \in \mathcal{C}^\theta$. Denote also $N = \mathrm{tr}(g)$ and $M = \chi(\mathcal{F}_0, F(\mathcal{F}_0)) = \chi(v_0, gv_0)$.*

(a) *If $M \geq N - 1$ then the sequence $(F^n(\mathcal{F}_0), n \in \mathbb{Z})$ in $\mathcal{C}^\theta$ is ample.*
(b) *If $0 < M < N - 1$ then the algebra $A_{F,\mathcal{F}_0}$ is not finitely generated, hence, the sequence $(F^n(\mathcal{F}_0))$ is not ample.*

Proof. Below we will denote the coordinates of a vector $v \in \mathbb{R}^2$ by $(\deg(v), \mathrm{rk}(v))$. Let us denote $\mathcal{F}_n = F^n(\mathcal{F}_0)$, $v_n = v_{\mathcal{F}_n} = g^n v_0$. By Proposition 2.6 one has $F(\mathcal{C}^\theta) = \mathcal{C}^\theta[m]$ for some $m \in \mathbb{Z}$. Hence, our assumption $F(\mathcal{F}_0) \in \mathcal{C}^\theta$ implies that $F(\mathcal{C}^\theta) = \mathcal{C}^\theta$. In particular, $\mathcal{F}_n \in \mathcal{C}^\theta$ for all $n \in \mathbb{Z}$.

(a) Assume first that $M \geq N$. Note that v_0 is not an eigenvector of g, so we can choose u in such a way that $\chi(u, v_0) > 0$. Then $\chi(u, v_n) > 0$ for all $n \in \mathbb{Z}$. Moreover, since $\chi(g^{-1}v_0, v_0) > 0$ and since u is an eigenvector of g^{-1} with the eigenvalue > 1, it follows that $\deg(v_n)/\mathrm{rk}(v_n)$ tends to θ as $n \to -\infty$. Observe also that all the vectors $\{v_{\mathcal{F}}, \mathcal{F} \in \mathcal{C}^\theta\}$ lie in the half-plane $H = \{v : \chi(u, v) > 0\}$ (since θ is irrational). It suffices to prove that for every $\mathcal{F} \in \mathcal{C}^\theta$ the following holds:

(i) $\mathrm{Hom}^1(\mathcal{F}_n, \mathcal{F}) = 0$ for $n \ll 0$;
(ii) the natural map $\mathrm{Hom}(\mathcal{F}_n, \mathcal{F}) \otimes \mathcal{F}_n \to \mathcal{F}$ is surjective for $n \ll 0$.

Moreover, it is enough to check these statements for stable $\mathcal{F}$. Also, for (ii) it is enough to prove that $L_{\mathcal{F}_n}(\mathcal{F})$ is in $\mathcal{C}^\theta$ for $n \ll 0$, where $L_{\mathcal{F}_n}$ is the left twist functor associated with $\mathcal{F}_n$. Since the vectors v_n lie in the half-plane H, $\chi(v_{n-1}, v_n) > 0$ and $\deg(v_n)/\mathrm{rk}(v_n) \to \theta$ as $n \to -\infty$, it follows that for every vector v in H one has $\chi(v_n, v) > 0$ for $n << 0$. Applying Lemma 2.7 to $\mathcal{F}_n$ and a stable object $\mathcal{F} \in \mathcal{C}^\theta$ we immediately derive (i). It remains to prove that for such $\mathcal{F}$ one has $L_{\mathcal{F}_n}(\mathcal{F}) \in \mathcal{C}^\theta$ for $n \ll 0$. Since $L_{\mathcal{F}_n}(\mathcal{F})$ is a stable object that fits into an exact triangle

$$\mathcal{F}[-1] \to L_{\mathcal{F}_n}(\mathcal{F}) \to \mathrm{Hom}(\mathcal{F}_n, \mathcal{F}) \otimes \mathcal{F}_n \to \mathcal{F},$$

it suffices to prove that $v_{L_{\mathcal{F}_n}(\mathcal{F})}$ belongs to H. But

$$v_{L_{\mathcal{F}_n}(\mathcal{F})} = \chi(\mathcal{F}_n, \mathcal{F}) v_n - v_{\mathcal{F}} = \chi(v_n, v_{\mathcal{F}}) v_n - v_{\mathcal{F}},$$

so

$$\chi(u, v_{L_{\mathcal{F}_n}(\mathcal{F})}) = \chi(v_n, v)\chi(u, v_n) - \chi(u, v),$$

where $v = v_{\mathcal{F}}$.

Let $r < 1$ be the eigenvalue of g corresponding to u and let u' be the eigenvector corresponding to r^{-1}. Rescaling u and u' we can assume that $v_0 = u + u'$. Then $v_n = r^n u + r^{-n} u'$, $\chi(u, v_n) = r^{-n}\chi(u, u') > 0$ and

$$\chi(v_n, v) = r^n \chi(u, v) + r^{-n}\chi(u', v).$$

It follows that

$$\chi(u, v_{L_{\mathcal{F}_n}(\mathcal{F})}) = (\Delta - 1)\chi(u, v) + r^{-2n}\Delta\chi(u', v),$$

where $\Delta = \chi(u, u')$. Since $\chi(u, v) > 0$, this quantity is positive for $n \ll 0$ provided that $\Delta > 1$. But

$$\Delta = \frac{\chi(v_0, gv_0)}{r^{-1} - r} \geq \frac{r + r^{-1}}{r^{-1} - r} > 1,$$

where we used our assumption $M \geq N$.

Now let us consider the case $M = N - 1$. The condition (i) is still satisfied, however (ii) has to be replaced by a weaker condition. For every $\mathcal{F} \in \mathcal{C}^\theta$ and $n \in \mathbb{Z}$ let us set

$$T_n \mathcal{F} := \mathrm{coker}(\mathrm{Hom}(\mathcal{F}_n, \mathcal{F}) \otimes \mathcal{F}_n \to \mathcal{F}).$$

It suffices to prove that for every $\mathcal{F}$ we have

$$T_{-n+m} \cdots T_{-1+m} T_m \mathcal{F} = 0$$

for some $m \in \mathbb{Z}$ and some $n > 0$. As before we can assume that $\mathcal{F}$ is a stable object in $\mathcal{C}^\theta$. Using the action of F we can reduce ourselves to the case when the vector $v = v_\mathcal{F}$ satisfies $\chi(v_0, v) > 0$. In this case we will show that $T_{-n} \cdots T_{-1} T_0 \mathcal{F} = 0$ for some $n > 0$. By Lemma 2.7, we have $\mathrm{Hom}^1(\mathcal{F}_0, \mathcal{F}) = 0$, so there is an exact triangle

$$L_{\mathcal{F}_0}(\mathcal{F}) \to \mathrm{Hom}(\mathcal{F}_0, \mathcal{F}) \otimes \mathcal{F}_0 \to \mathcal{F} \to L_{\mathcal{F}_0}(\mathcal{F})[1].$$

Let $h_i \in \mathrm{SL}_2(\mathbb{Z})$ be the matrix corresponding to the functor $L_{\mathcal{F}_i}[1] : D^b(X) \to D^b(X)$. We claim that if $v_{L_{\mathcal{F}_0}(\mathcal{F})[1]} = h_0(v) \neg \in H$ then $T_0 \mathcal{F} = 0$. Indeed, since $L_{\mathcal{F}_0}(\mathcal{F})$ is a stable object, this would imply that $L_{\mathcal{F}_0}(\mathcal{F}) \in \mathcal{C}^\theta$, so the map $\mathrm{Hom}(\mathcal{F}_0, \mathcal{F}) \otimes \mathcal{F}_0 \to \mathcal{F}$ is surjective. Otherwise, we have $v_{L_{\mathcal{F}_0}(\mathcal{F})[1]} \in H$ which implies that $L_{\mathcal{F}_0}(\mathcal{F})[1]$ belongs to $\mathcal{C}^\theta$, the map $\mathrm{Hom}(\mathcal{F}_0, \mathcal{F}) \otimes \mathcal{F}_0 \to \mathcal{F}$ is injective with the cokernel

$$T_0 \mathcal{F} \simeq L_{\mathcal{F}_0}(\mathcal{F})[1],$$

and $v_{T_0 \mathcal{F}} = h_0(v)$. Continuing to argue in this way we see that it is enough to show the existence of $n > 0$ such that $h_{-n} \cdots h_{-1} h_0(v) \neg \in H$. Using the formula $v_{L_{\mathcal{F}_0}(\mathcal{F})[1]} = v - \chi(v_0, v) v_0$ we can write the matrix of h_0 with respect to the basis (u, u'):

$$h_0 = \begin{pmatrix} 1 + \Delta & -\Delta \\ \Delta & 1 - \Delta \end{pmatrix},$$

where $\Delta = \chi(u, u')$. Similarly,

$$h_{-i} = \begin{pmatrix} r^{-i} & 0 \\ 0 & r^i \end{pmatrix} h_0 \begin{pmatrix} r^i & 0 \\ 0 & r^{-i} \end{pmatrix}.$$

Therefore,

$$h_{-n} \cdots h_{-1} h_0 = \begin{pmatrix} r^{-n+1} & 0 \\ 0 & r^{n-1} \end{pmatrix} S^{n+1},$$

where

$$S = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} h_0 = \begin{pmatrix} r(1 + \Delta) & -r\Delta \\ r^{-1}\Delta & r^{-1}(1 - \Delta) \end{pmatrix}.$$

But $\det(S) = 1$ and

$$\mathrm{tr}(S) = r(1 + \Delta) + r^{-1}(1 - \Delta) = N + (r - r^{-1})\Delta = N - M = 1.$$

Hence, $S^2 - S + 1 = 0$ and therefore $S^3 = -1$. It follows that $h_{-2} h_{-1} h_0(v)$ is not in H which finishes the proof.

(b) Let $A = A_{F, \mathcal{F}_0}$. For a graded right A-module M and $n \in \mathbb{Z}$ let us set

$$T_n M := \mathrm{coker}(M_n \otimes A(-n) \to M),$$

where $A(-n)$ is a free A-module with $A(-n)_i = A_{i-n}$. To show that the algebra A is not finitely generated we have to show that for all $n \geq 1$ one has

$$T_n T_{n-1} \cdots T_1 A_{\geq 1} \neq 0,$$

where $A_{\geq 1} = \oplus_{i \geq 1} A_i$.

For every $\mathcal{F} \in \mathcal{C}^\theta$ and $n \in \mathbb{Z}$ we set

$$\Gamma_{\geq n}(\mathcal{F}) = \bigoplus_{m \geq n} \operatorname{Hom}(\mathcal{F}_{-m}, \mathcal{F}).$$

This space has a natural structure of a graded right A-module. For example, we have $\Gamma_{\geq n}(\mathcal{F}_{-n}) = A(-n)$. Now we claim that it is enough to prove that $h_{-n} h_{-n+1} \cdots h_{-1}(v_0)$ is a nonzero vector in H for all $n \geq 1$, where we use the notation from the proof of part (a). Indeed, as we have seen above this would imply that for every $n \geq 1$ the object $T_{-n} T_{-n+1} \cdots T_1 \mathcal{F}_0$ is stable and that we have exact sequences

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathcal{F}_{-n}, T_{-n+1} \cdots T_1 \mathcal{F}_0) \otimes \mathcal{F}_{-n} \to T_{-n+1} \cdots T_1 \mathcal{F}_0 \\ \to T_{-n} T_{-n+1} \cdots T_1 \mathcal{F}_0 \to 0 \end{aligned}$$

in $\mathcal{C}^\theta$. Using these exact sequences and Lemma 2.7 one can easily see that

$$\Gamma_{\geq n+1}(T_{-n} T_{-n+1} \cdots T_1 \mathcal{F}_0) \simeq T_n T_{n-1} \cdots T_1 A_{\geq 1}.$$

Again applying Lemma 2.7 we conclude that this space is not zero for every $n \geq 1$ which proves our claim.

To prove that $h_{-n} \cdots h_{-1}(v_0)$ is a nonzero vector in H it suffices to show that $S^n(\mathbb{R}_{>0} u + \mathbb{R}_{>0} u') \subset H$ for all $n \geq 1$. Note that $\operatorname{tr}(S) = N - M \geq 2$, so S has real positive eigenvalues. Since $\chi(u, Su) = r^{-1}\Delta^2 > 0$, it is enough to prove that here exists an eigenvector of S of the form $-xu + u'$, where $x > 0$. Equivalently, the equation

$$\chi(-xu + u', S(-xu + u')) = r^{-1}\Delta x^2 + [r(1+\Delta) - r^{-1}(1-\Delta)]x + r\Delta = 0$$

should have a positive root. For this two inequalities should hold: $D = b^2 - 4\Delta^2 \geq 0$ and $b < 0$, where $b = r(1+\Delta) - r^{-1}(1-\Delta)$. But

$$b = r - r^{-1} + N\Delta = \frac{NM}{r^{-1} - r} - r^{-1} + r = \frac{NM - N^2 + 4}{r^{-1} - r} < 0$$

since $M \leq N - 2$ and $N > 2$. Finally,

$$\begin{aligned} D &= (r - r^{-1} + N\Delta)^2 - 4\Delta^2 = N^2 - 4 - 2NM + (N^2 - 4)\Delta^2 \\ &= N^2 - 4 - 2NM + M^2 \geq 0 \end{aligned}$$

since $N - M \geq 2$. □

Remark 4.3. If the equivalent conditions of the above theorem are satisfied then we also have $F^n(\mathcal{F}_0) \in \mathcal{C}^{\theta'}$ for all n, where $(\theta', 1)$ is the eigenvector of g corresponding to the eigenvalue > 1. The sequence $(F^n(\mathcal{F}_0))$ is not ample in $\mathcal{C}^{\theta'}$, since there are objects $\mathcal{F} \in \mathcal{C}^{\theta'}$ with $\operatorname{Hom}^1(F^n(\mathcal{F}_0), \mathcal{F}) \neq 0$ and $\operatorname{Hom}(F^n(\mathcal{F}_0), \mathcal{F}) = 0$ for all $n \ll 0$. Nevertheless, we still have an equivalence of the derived category of $\mathcal{C}^{\theta'}$ with the derived category of cohproj $A_{F,\mathcal{F}}$, since both categories are equivalent to $D^b(X)$. It would be interesting to find a general framework for this kind of equivalences associated with nonample sequences.

Corollary 4.4. *Let $(F, \mathcal{F})$ be a pair satisfying the equivalent conditions of Proposition 3.3 and let $\pi(F) = g \in \mathrm{SL}_2(\mathbb{Z})$, $N = \mathrm{tr}(g)$, $M = \chi(\mathcal{F}, F(\mathcal{F}))$. Then the algebra $A_{F,\mathcal{F}}$ is finitely generated if and only if $M \geq N - 1$.*

Proof. If g has distinct eigenvalues then this follows from Theorem 4.2. Now assume that g is unipotent (so that $N = 2$). Then the statement reduces to the case when F is a composition of the tensoring by a line bundle L with an automorphism of X. In this case we can assume that $\mathcal{F} \in \mathrm{Coh}\, X$. Since $M \geq 1$ it follows that $\mathcal{F}$ is a vector bundle and $\deg(L) \geq 1$. It easy to see that in this case the sequence $(F^n(\mathcal{F}))$ is ample, hence, the algebra $A_{F,\mathcal{F}}$ is finitely generated. □

4.2. Projectivity of $\mathcal{C}^\theta$

Now we can show that every category $\mathcal{C}^\theta$, where θ is a quadratic irrationality, can be described as a "noncommutative Proj".

Theorem 4.5. *For every quadratic irrationality $\theta \in \mathbb{R}$ there exists an autoequivalence $F : D^b(X) \to D^b(X)$ preserving $\mathcal{C}^\theta$ and a stable object $\mathcal{F} \in \mathcal{C}^\theta$ such that the sequence $(F^n\mathcal{F}, n \in \mathbb{Z})$ is ample. Hence, the corresponding algebra $A_{F,\mathcal{F}}$ is right coherent and $\mathcal{C}^\theta \simeq \mathrm{cohproj}\, A_{F,\mathcal{F}}$.*

Proof. Let $\alpha\theta^2 + \beta\theta + \gamma = 0$ be the equation satisfied by θ, where $\alpha, \beta, \gamma \in \mathbb{Z}$, $\alpha > 0$. Consider the ring $R = \mathbb{Z}[\alpha\theta] \subset \mathbb{Q}(\theta)$. Then R is contained in $\mathbb{Z} + \mathbb{Z}\theta$ and $R(\mathbb{Z} + \mathbb{Z}\theta) \subset \mathbb{Z} + \mathbb{Z}\theta$. Let $r \in R^*$ be a unit such that $0 < r < 1$ (such a unit always exists). Then the multiplication by r induces an invertible operator on $\mathbb{Z} + \mathbb{Z}\theta$ with determinant equal to $\mathrm{Nm}(r) = 1$. Hence, we have $r = c\theta + d$, $r\theta = a\theta + b$ for some

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Then $u = (\theta, 1) \in \mathbb{R}^2$ is an eigenvector of g corresponding to the eigenvalue r. We claim that there exists a primitive vector $v \in \mathbb{Z}^2$ such that $\chi(u, v) > 0$ and $\chi(v, gv) \geq \mathrm{tr}(g)$. Indeed, let $u' \in \mathbb{R}^2$ be an eigenvector of g corresponding to the eigenvalue r^{-1} and such that $\chi(u, u') > 0$. We can find a primitive vector $v \in \mathbb{Z}^2$ such that $v = xu + yu'$, where $x > 0$, $y > 0$, and $xy \geq (r + r^{-1})/(r^{-1} - r)\chi(u, u'))$. Then $\chi(u, v) = y\chi(u, u') > 0$ and

$$\chi(v, gv) = \chi(xu + yu', xru + yr^{-1}u') = xy(r^{-1} - r)\chi(u, u') \geq r + r^{-1}$$

as required. It remains to choose an autoequivalence F with $\pi(F) = g$ such that F preserves $\mathcal{C}^\theta$ and an object $\mathcal{F} \in \mathcal{C}^\theta$ with $v_{\mathcal{F}} = v$, and then apply Theorem 4.2. □

Finally, we are going to show that the category $\mathcal{C}^\theta$ for arbitrary $\theta \in \mathbb{R}$ can be represented in the form cohproj A for some coherent $\mathbb{Z}$-algebra A. Recall that the notion of $\mathbb{Z}$-algebra is a natural generalization of the notion of graded algebra (see [4,13]): such an algebra is equipped with a decomposition $A = \oplus_{i\le j} A_{i,j}$ and the case of a graded algebra corresponds to $A_{i,j} = A_{j-i}$. As in the case of real multiplication considered above, it is enough to construct an ample sequence $(\mathcal{F}_n, n \in \mathbb{Z})$ of objects in $\mathcal{C}^\theta$, however, not necessarily of the form $\mathcal{F}_n = F^n(\mathcal{F}_0)$ for some autoequivalence F. Then the main theorem of [13] will give an equivalence $\mathcal{C}^\theta \simeq$ cohproj A, where A is the $\mathbb{Z}$-algebra associated with the sequence $(\mathcal{F}_n)$, so that $A_{i,j} = \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}_j)$. The construction of the following theorem provides plenty of ample sequences in $\mathcal{C}^\theta$.

Theorem 4.6. *For every $\theta \in \mathbb{R}$ there exists an ample sequence $(\mathcal{F}_n, n \in \mathbb{Z})$ in $\mathcal{C}^\theta$ such that all the objects $\mathcal{F}_n$ are stable.*

Proof. Clearly, it suffices to consider the case when θ is irrational. Recall that all vectors $v_{\mathcal{F}}$ for $\mathcal{F} \in \mathcal{C}^\theta$ belong to the half-plane $H = H_\theta = \{(x_1, x_2) | x_1 - \theta x_2 > 0\} \subset \mathbb{R}^2$. Moreover, for every primitive vector $v \in H \cap \mathbb{Z}^2$ there exists a stable object $\mathcal{F} \in \mathcal{C}^\theta$ with $v_{\mathcal{F}} = v$. Now let us choose a sequence of primitive vectors $v_n = (d_n, r_n) \in H \cap \mathbb{Z}^2$ such that $r_n > 0$ for $n \ll 0$ and $\lim_{n\to -\infty} \mu_n = \theta$, where $\mu_n = d_n/r_n$. In other words, we want the ray $\mathbb{R}_{\ge 0} v_n$ to approach $\mathbb{R}_{\ge 0}(\theta, 1)$ as $n \to -\infty$. Note that since $d_n - \theta r_n > 0$ we necessarily have $\mu_n > \theta$ for $n \ll 0$. In addition we can make this choice in such a way that for all $n \ll 0$ one has $\mu_n - \theta \ge r_n^{-1}$. Indeed, we can first choose r_n for $n \ll 0$ to be a sequence of prime numbers such that $\lim_{n\to -\infty} r_n = +\infty$. Then after picking any sequence d_n such that $\lim_{n\to -\infty} d_n/r_n = \theta$ we can change d_n by $d_n + 1$ if necessary to make $\mu_n - \theta \ge r_n^{-1}$ for $n \ll 0$. Since θ is not an integer and d_n/r_n tends to θ as $n \to -\infty$, such a change will leave d_n prime to r_n.

Now we claim that if $(\mathcal{F}_n, n \in \mathbb{Z})$ is any sequence of stable objects in $\mathcal{C}^\theta$ with $v_{\mathcal{F}_n} = v_n$ then conditions (i) and (ii) from the proof of Theorem 4.2 are satisfied for every stable $\mathcal{F} \in \mathcal{C}^\theta$, and therefore the sequence $(\mathcal{F}_n)$ is ample. Indeed, condition (i) follows from Lemma 2.7 since for every $v \in H$ one has $\chi(v_n, v) > 0$ for $n << 0$. Arguing in the same way as in the proof of Theorem 4.2 we conclude that it is enough to prove that for every $v \in H \cap \mathbb{Z}^2$ one has $\chi(v_n, v) v_n - v \in H$ for $n \ll 0$. Let $v = (d, r)$. Then we have to show that

$$(dr_n - d_n r)d_n - d - \theta[(dr_n - d_n r)r_n - r] > 0$$

for $n \ll 0$. Assume first that $r \ne 0$ and set $\mu = d/r$. Then the above inequality can be rewritten as

$$rr_n^2(\mu - \mu_n)(\mu_n - \theta) > r(\mu - \theta).$$

Note that $r(\mu - \mu_n) = \chi(v_n, v)/r_n > 0$ for $n \ll 0$. Hence, our inequality for $n \ll 0$ is equivalent to

$$r_n^2(\mu_n - \theta) > \frac{\mu - \theta}{\mu - \mu_n}.$$

But this follows from the condition that $\mu_n - \theta \geq r_n^{-1}$ since $r_n \to +\infty$ as $n \to -\infty$. Similar argument works in the case $r = 0$. $\square$

Acknowledgements

This work was partially supported by NSF grant DMS-0070967.

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